THE ELECTRIC FIELD IN A MAGNETO-HYDRODYNAMIC CHANNEL CONTAINING A MOVING MEDIUM WITH VARIABLE CONDUCTIVITY

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The usual method of approximate calculation of electric fields [1] has already been widely applied to the solution of various steady problems, in particular, to the problem of a plane magnetohydrodynamic channel, through which flows a medium possessing variable conductivity along the channel [2]. The variation of the electrical quantities with time have been studied by similar methods only in the case of periodic variation of the magnetic field [3] or for an arbitrary law of motion of the medium [4]. In connection with certain applications [5] there is interest in the study of channels in which a nonconducting gas carries with it separate conducting clots. The solution is derived below for one of the simple problems concerning the distribution of electric field in the channel with allowance for the periodic variation of conductivity with respect to time and the longitudinal coordinate.

1. We shall consider, on the basis of reasoning in [1], the motion of an isotropic conducting medium in a rectangular channel (Fig.1) placed in a magnetic field. We shall suppose that the velocity of the stream $\mathbf{v} = \mathbf{e}_x U$ is everywhere constant, the magnetic field $\mathbf{B}(x, z) = \mathbf{e}_x B_x + \mathbf{e}_z B_z$ is independent of time, and that the conductivity of the medium is a periodic function of time t and of the coordinate x, represented in the form

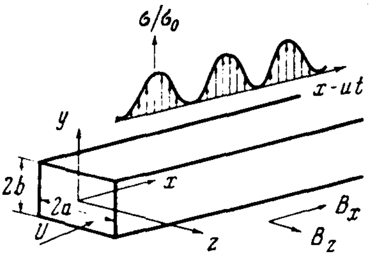

Fig. 1

$$\sigma = \sigma_0 \psi(x - Ut) = \sigma_0 \psi(x + \lambda - Ut) \qquad (1.1)$$

If the magnetic Reynolds number R_m is small, while the ratio of the transverse dimension of the channel to λ is a quantity of smaller order than R_m^{-1}, then the induced magnetic field and its derivative with respect to time can be neglected in Ohm's law and Maxwell's equations. Then the electric field is quasi-steady and its potential φ will depend on the time as well as on a parameter.

For the potential φ and the current j, averaged with respect to z, assuming the walls of the channel to be nonconducting, we can write

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\psi'}{\psi}\frac{\partial\varphi}{\partial x}=0, \qquad j_x=-\sigma\frac{\partial\varphi}{\partial x}, \qquad j_y=-\sigma\left(\frac{\partial\varphi}{\partial y}+\frac{UB}{c}\right) \tag{1.2}$$

where $B = B_0 f(x)$ is the distribution of the z-component of the magnetic field averaged with respect to z. We shall assume moreover that $f(x)$, together with its derivatives up to the second order, tend to zero as $|x|\to\infty$.

The boundary conditions for φ have the form

$$\frac{\partial\varphi}{\partial y}=-\frac{UB}{c} \quad \text{for } y=\pm\delta, \qquad \nabla\varphi\to 0 \quad \text{for } |x|\to\infty \tag{1.3}$$

We introduce the auxiliary potential $\Phi=\varphi+UBy/c$, satisfying the inhomogeneous equation of type (1.2) and the homogeneous boundary conditions. Taking as the units of length, time, potential and current the quantities $\delta,\ \lambda/U,\ UB_0\delta/c,\ \sigma_0 UB_0/c$, respectively, we obtain instead of (1.2) and (1.3)

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+F\frac{\partial\Phi}{\partial x}=(f''+Ff')\,y, \quad j_x=-\psi\frac{\partial\Phi}{\partial x}, \quad j_y=-\psi\frac{\partial\Phi}{\partial y} \tag{1.4}$$

$$\frac{\partial\Phi}{\partial y}=0 \quad \text{for } y=\pm 1, \qquad \nabla\Phi\to 0 \quad \text{for } |x|\to\infty \tag{1.5}$$

$$(\psi=\psi(\xi), \quad F=\psi'/\psi, \quad \xi=x-Vt, \quad V=\lambda/\delta)$$

The solution of the problem will be sought in the form of the expansion

$$\Phi=\sum_{k=1}^{\infty}\Phi_k(x,t)\sin\alpha_k y, \qquad \alpha_k=\pi(k-{}^1/_2) \tag{1.6}$$

By the usual method from (1.4) we obtain

$$\Phi_k''+F\Phi_k'-\alpha_k^2\Phi_k=\omega_k(f''+Ff'), \qquad \omega_k=2(-1)^{k+1}\alpha_k^{-2}$$
$$\Phi_k\to 0 \qquad \text{for } |x|\to\infty \tag{1.7}$$

The solution of the problem (1.7) can be expressed by means of tabulated functions only for isolated particular cases of the function $\psi(\xi)$, one of which will be considered below.

2. Suppose that the conductivity of the medium is determined by Formula $\psi=\cos^2\beta\xi$ and then $V=\pi/\beta$. Equation (1.7), taking the form

$$\Phi_k''-2\Phi_k'\beta\tan\beta\xi-\alpha_k^2\Phi_k=\omega_k(f''-2f'\beta\tan\beta\xi) \tag{2.1}$$

has the general solution when $\alpha_k\neq\beta$ (2.2)

$$\Phi_k=\frac{1}{\cos\beta\xi}\left\{e^{\gamma_k\beta\xi}\left[C_k-\int_a^x\Omega_k(s,t)\,e^{-\gamma_k\beta\zeta}ds\right]+e^{-\gamma_k\beta\xi}\left[D_k+\int_b^x\Omega_k(s,t)\,e^{\gamma_k\beta\zeta}ds\right]\right\}$$

$$\Omega_k=-\frac{\omega_k}{2\gamma_k\beta}\left[f''(s)\cos\beta\zeta-2f'(s)\beta\sin\beta\zeta\right] \qquad (\zeta=s-Vt, \ \gamma_k^2=\alpha_k^2\beta^{-2}-1)$$

It is easy to see that when $f'\not\equiv 0$ and the arbitrary choice of the constants C_k, D_k, a and b, the solution (2.2) does not satisfy the condition $\Phi_k\to 0$ when $|x|\to\infty$, since it has poles at the points $x_n=Vt+\pi(2n+1)/2\beta$ for all the values $n=0,\pm 1,\pm 2,\pm 3,\ \ldots$ (with the exception, perhaps, of certain of these values). Accordingly, when $f \not\equiv \text{const}$ there does not exist a solution of the problem (1.7), continuous and bounded for $x \in (-\infty, \infty)$, if $F(\xi) \sim \tan \beta\xi$. It is essential to assume that the solution will not exist also for a different choice of $F(\xi)$, if $F(\xi)$ has at the points ξ_n singularities of the type $(\xi - \xi_n)^{-1-\alpha}$, $\alpha > 0$.

On the other hand, we can attempt to construct a bounded but discontinuous solution of the problem (1.7), considering separately each interval $[x_{n-1}, x_n]$ and choosing therein different values of the constants C_k, D_k from the condition of boundedness of the solution at the ends of the interval.

Suppose that $x \in [x_{n-1}, x_n]$. Let us set $a = x_n$, $b = x_{n-1}$ in (2.2) and introduce the notation

$$G_{nk}(x, t) = \int_{x_n}^{x} \Omega_k(s, t)\, e^{-\gamma_k \beta\zeta} ds, \qquad H_{nk}(x, t) = \int_{x_{n-1}}^{x} \Omega_k(s, t)\, e^{\gamma_k \beta\zeta} ds \tag{2.3}$$

The functions G_{nk}, H_{nk} are defined and continuous in the interval $[x_{n-1}, x_n]$. Setting (2.2) in the form

$$\Phi_k = \frac{1}{\cos\beta\xi}\left[(C_{nk} - G_{nk})\, e^{\gamma_k \beta\xi} + (D_{nk} + H_{nk})\, e^{-\gamma_k \beta\xi}\right], \qquad x \in [x_{n-1}, x_n] \tag{2.4}$$

let us require that the numerator vanishes when $x = x_{n-1}$ and $x = x_n$. It is not difficult to see that for this it is necessary to take

$$C_{nk} = -(2\sinh \pi\gamma_k)^{-1}\left[G_{nk}(x_{n-1})\, e^{-\gamma_k \pi} + H_{nk}(x_n)\, e^{-2\gamma_k \pi n}\right]$$

$$D_{nk} = (2\sinh \pi\gamma_k)^{-1}\left[G_{nk}(x_{n-1})\, e^{2\gamma_k \pi n} + H_{nk}(x_n)\, e^{-\gamma_k \pi}\right] \tag{2.5}$$

Now $\Phi_k(x_{n-1})$, $\Phi_k(x_n)$ are bounded, since both the numerator and the denominator of (2.4) have first order zeros at these points. In addition to this, the limiting values of $\Phi_k(x_n - 0)$ and $\Phi_k(x_n + 0)$ may not coincide if we do not impose special limitations on the field distribution $f(s)$.

We shall show that the solution (2.4), (2.5), which is discontinuous in the general case, satisfies the conditions when $|x| \to \infty$, which for a fixed value of time t can be written in the form

$$\lim_{|n|\to\infty} \Phi_k\{x \in [x_{n-1}, x_n]\} = 0 \tag{2.6}$$

In fact, from Formula (2.3) there follow the inequalities

$$|G_{nk}(x)| \leqslant \frac{|\omega_k|\pi}{2\gamma_k\beta^2} e^{-\gamma_k \beta\xi}(M_1 + 2\beta M_2), \qquad |H_{nk}(x)| \leqslant \frac{|\omega_k|\pi}{2\gamma_k\beta^2} e^{\gamma_k \beta\xi}(M_1 + 2\beta M_2)$$

$$M_1 = \max|f''(s)|, \quad M_2 = \max|f'(s)|, \quad s \in [x_{n-1}, x_n] \tag{2.7}$$

Hence, taking into consideration (2.5), we find that when $x \in [x_{n-1}, x_n]$ and $n \to \infty$, the function $|C_{nk} - G_{nk}(x)$ tends to zero more quickly than $e^{-\gamma_k \beta x}$, whilst $|D_{nk} + H_{nk}(x)|$ is bounded or tends to infinity not more quickly than $e^{-\gamma_k \beta x}$. This deduction is based on the assumption concerning the decrease of $M_{1,2}$ as $s \to \infty$, which was stipulated in Section 1. Referring now to (2.4), we see that both terms of the numerator tend to zero like $M_1 + 2\beta M_2$ as $n \to \infty$. Therefore $\Phi_k \to 0$ also as $n \to \infty$, if $x \in (x_{n-1}, x_n)$.

The decrease in Φ_k at the ends of the segments when $n \to \infty$ is easy to prove, if we calculate first of all $\Phi_k(x_n \pm 0)$, applying l'Hôpital's rule and the relations (2.3),(2.5)

$$\lim_{x \to x_n - 0} \Phi_k = (-1)^n \gamma_k \{[D_{nk} + H_{nk}(x_n)]\, e^{-\gamma_k \beta \xi_n} - C_{nk} e^{\gamma_k \beta \xi_n}\}$$

$$(\xi_n = x_n - Vt) \tag{2.8}$$

$$\lim_{x \to x_n + 0} \Phi_k = (-1)^n \gamma_k \{D_{n+1,k} e^{-\gamma_k \beta \xi_n} - [C_{n+1,k} - G_{nk}(x_n)]\, e^{\gamma_k \beta \xi_n}\}$$

Similar considerations apply also when $n \to -\infty$. Accordingly, Formulas (2.3),(2.5) define a bounded solution of the problem (1.7), which satisfies the conditions at infinity and is continuous in each interval (x_{n-1}, x_n).

If we consider the behavior of Φ_k at a fixed point x, then to the continuous increase of time t there will correspond a decrease in the integer argument n (with the notation in the form (2.4)). The interval of time, in the course of which the point x belongs to the segment $[x_{s-1}, x_s]$, is determined by the inequalities

$$\frac{1}{V}\left[x - \frac{\pi}{2\beta}(2s+1)\right] \leqslant t \leqslant \frac{1}{V}\left[x - \frac{\pi}{2\beta}(2s-1)\right]$$

and for these values of t the function Φ_k is given by Formula (2.4) with $n = s$. For the succeeding segment $[x_{s-2}, x_{s-1}]$

$$\frac{1}{V}\left[x - \frac{\pi}{2\beta}(2s-1)\right] \leqslant t \leqslant \frac{1}{V}\left[x - \frac{\pi}{2\beta}(2s-3)\right]$$

and in (2.4) one must now set $n = s - 1$, and so on. If x and t are given, then from these inequalities we determine the number of the segment to which the point x belongs at the instant t. With the elapse of time $T = \pi/\beta V = 1$ the position of the point x in relation to the ends of the segment is repeated and, as the calculations show,

$$\Phi_k(x, t)\,|_{n=s} = \Phi_k(x, t+1)\,|_{n=s-1}$$

Hence it follows that the solution constructed for Φ_k and, evidently, for Φ and φ, is periodic with respect to time.

At the points $x = x_n$ the functions Φ_k undergo discontinuities, wherein the magnitude of the jump is determined according to (2.5) and (2.8) by Formula

$$\{\Phi_k\}_{x=x_n} = \Phi_k(x_n + 0) - \Phi_k(x_n - 0) =$$

$$= \frac{2\gamma_k(-1)^n}{\sinh \pi\gamma_k} \int_0^{\pi/\beta} [\Omega_k(x_n - \tau) + \Omega_k(x_n + \tau)] \sinh \gamma_k(\beta\tau - \pi)\, d\tau \tag{2.9}$$

The continuity in the potential φ and the component of the electric field $E_y = -\partial\varphi/\partial y$, tangent to the moving line $x = x_n$, are, respectively, expressed (in dimensionless quantities) by Formulas

$$\{\varphi\} = -y\,\{f\} + \sum_{k=1}^{\infty} \{\Phi_k\} \sin \alpha_k y, \qquad \{E_y\} = \{f\} - \sum_{k=1}^{\infty} \{\Phi_k\}\, \alpha_k \cos \alpha_k y \tag{2.10}$$

The discontinuity in the magnetic field $\{f\}$ in reality does not make a contribution to $\{\varphi\}$ and $\{E_y\}$: if we isolate from $\{\Phi_k\}$ the part connected with the discontinuity $\{f\}$, then after some calculation Formulas (2.10) take the form

$$\{\varphi\} = \sum_{k=1}^{\infty} \{\Phi_k\}^* \sin \alpha_k y, \qquad \{E_y\} = -\sum_{k=1}^{\infty} \{\Phi_k\}^* \alpha_k \cos \alpha_k y \qquad (2.11)$$

Here $\{\Phi_k\}^*$ denotes the result of substituting in (2.10) only the continuous part (*) of $f(x)$. Consequently, $\{\varphi\}$ and $\{E_y\}$ vanish together with $\{\Phi_k\}^*$. A sufficient condition for this is, for example, that the function $f(x_n+\tau)$ be continuous and an even function of τ when $\tau \in [-\pi/\beta, \pi/\beta]$, since then $\Omega_k(x_n+\tau)$ in this interval is an odd function of τ.

Direct calculation shows moreover that

$$\lim_{x \to x_n \pm 0} \frac{\partial \Phi_k}{\partial x} = \omega_k f'(x_n \pm 0), \qquad \lim_{x \to x_n \pm 0} \frac{\partial \varphi}{\partial x} = 0 \qquad (2.12)$$

i.e. $\partial\varphi/\partial x$ is continuous at the points $x = x_n$. It turns out also that all the successive derivatives with respect to x are finite.

Accordingly, the constructed solution is similar to the potential of a double layer on the lines $x = x_n$: the normal derivative on them is finite and continuous, whilst the potential itself and the tangential derivative have discontinuities.

3. The appearance of a discontinuity of the tangential component of the electric field may at first sight appear paradoxical and contradictory to the usual assumptions of electrodynamics concerning the continuity of E_τ at the interfaces between media (**).

In actual fact, this effect is connected with the assumption concerning the existence of infinitely thin layers with nonzero electrical resistence. It is therefore appropriate to consider a simple model permitting study of the discontinuities in E_τ as the result of a certain limiting transition.

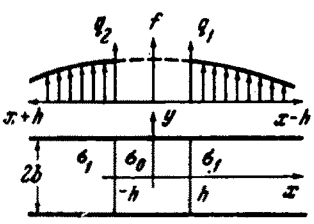

Fig. 2

Suppose, for example, that the conductivity of the fluid in the channel is equal to σ_1 when $|x| > h$ and σ_0 when $|x| < h$, where σ_0, σ_1 are constants (Fig.2). Then, retaining the assumptions stipulated in Section 1, the corresponding problem for finding the potential has in dimensionless variables the form [2]

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \quad \text{for } x \in (-\infty, h),\ (-h, h),\ (h, \infty)$$

$$\{\varphi\} = 0, \quad \left\{\psi \frac{\partial\varphi}{\partial x}\right\} = 0 \quad \text{for } x = \pm h \qquad (3.1)$$

$$\nabla\varphi \to 0 \quad \text{for } |x| \to \infty, \quad \frac{\partial\varphi}{\partial y} = -f(x) \quad \text{for } y = \pm 1$$

The solution of this problem is given by the series

$$\varphi = -y f(x) + \sum_{k=1}^{\infty} \Psi_k(x) \sin \alpha_k y$$

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*) If the function $f(x)$ is represented in the form

$$f(x) = f_0(x) + \Sigma a_i \eta(x - b_i), \quad a_i = \text{const}, \ b_i = \text{const}$$

where η is the Heaviside unit function, $f_0(x)$ is a continuous function, then $f_0(x)$ is called the continuous part of $f(x)$.

**) Other problems can also be cited where this condition is not fulfilled. For example, in a study of the electric field in a channel with dielectric partitions [6] a discontinuity of E_τ was found along the partitions.

$$\Psi_k = \begin{cases} \Psi_{1k} = C_{1k}e^{\alpha_k x} + L_k(x, -\infty) & \text{for } x \in (-\infty, h) \\ \Psi_{2k} = C_{2k}e^{\alpha_k x} + D_{2k}e^{-\alpha_k x} + L_k(x, 0) & \text{for } x \in (-h, h) \\ \Psi_{3k} = D_{3k}e^{-\alpha_k x} + L_k(x, \infty) & \text{for } x \in (h, \infty) \end{cases}$$

$$L_k(x, a) = \frac{\omega_k}{2\alpha_k}\left[e^{\alpha_k x}\int_a^x e^{-\alpha_k s} f''(s)\,ds - e^{-\alpha_k x}\int_a^x e^{-\alpha_k s} f''(s)\,ds\right]$$

The constants C_{ik}, D_{ik} are determined by extremely cumbersome formulas, only one of which will be reproduced here, ($\varepsilon = \sigma_0/\sigma_1$)

$$C_{2k} - D_{2k} = [2\alpha_k(\sinh\alpha_k h + \varepsilon\cosh\alpha_k h)^{-1}] \times$$
$$\times \{\alpha_k[L_k(h, \infty) - L_k(-h, -\infty) - L_k(h, 0) + L_k(-h, 0)] + \tag{3.2}$$
$$+ (\varepsilon - 1)[f'(h) + f'(-h)] + L_k'(h, \infty) + L_k'(-h, -\infty) - \varepsilon[L_k'(h, 0) + L_k'(-h, 0)]\}$$

enabling us to find the quatities (*)

$$V_k = \Psi_k(h) - \Psi_k(-h) = 2\sinh\alpha_k h(C_{2k} - D_{2k}) + L_k(h, 0) - L_k(-h, 0)$$
$$\left.\frac{\partial\Psi_k}{\partial x}\right|_{x=0} = \alpha_k(C_{2k} - D_{2k}) \tag{3.3}$$

We shall assume that $\varepsilon \ll 1$ and $h \ll 1$ then

$$V_k \approx -\frac{\alpha_k^2 h\omega_k}{\alpha_k h + \varepsilon}\left(\int_\infty^0 e^{-\alpha_k s} f(s)\,ds + \int_{-\infty}^0 e^{\alpha_k s} f(s)\,ds\right) + O(h^3), \qquad \left.\frac{\partial\Psi_k}{\partial x}\right|_{x=0} \approx \frac{V_k}{h} \tag{3.4}$$

The ratio $h/\varepsilon = R$ characterizes the electrical resistance of the layer $(-h, h)$. If as $h \to 0$, $\varepsilon \to 0$, the resistance R increases without limit or tends to a nonzero limit, then $\lim V_k$ is finite and nonzero (vanishing is possible, given a special choice of $f(s)$), whilst $(\partial\Psi_k/\partial x)_{x=0} \to \infty$. If, however, $R \to 0$, then $V_k \to 0$, and the behavior of $(\partial\Psi_k/\partial x)_{x=0}$ can then be different. We note also that when $R \to \text{const} \geqslant 0$ the current j_x at the section $x = 0$ also tends to a constant value, which decreases to zero with increase of the limiting value of R.

Accordingly, to the discontinuity Ψ_k at a certain line (i.e. the discontinuity of the potential φ) there corresponds an infinite growth of the normal derivative $\partial\varphi/\partial x$ at the internal points of the narrow layer with small conductivity, contracting to a line of discontinuity.

The condition of continuity of $\mathbf{E}_\tau$ in the general theory follows from the assumption of boundedness of E_n. From the foregoing example it follows that this assumption may not be fulfilled, and then there will exist a discontinuity in $\mathbf{E}_\tau$. The infinite growth of E_n may moreover, as for example in Section 2, not appear clearly in the problem, since the solution for the infinitely thin layer with small conductivity is not considered.

Formula (3.3), when $\varepsilon = 0$, takes the form

$$V_k = -\alpha_k\omega_k\left[\int_\infty^h e^{\alpha_k(h-s)} f(s)\,ds + \int_{-\infty}^{-h} e^{\alpha_k(h+s)} f(s)\,ds\right] \tag{3.5}$$

Suppose that the distribution of magnetic field when $x > h$ and $x < -h$ is determined only by the distance from the points $\pm h$, respectively, i.e.

$$f(x) = \begin{cases} q_1(x - h) & \text{for } x \geqslant h \\ q_2(x + h) & \text{for } x \leqslant -h \end{cases}$$

*) In (3.2) and the remainder of Section 3 it will be assumed that the function $f(x)$ is continuous everywhere on the axis.

Then from (3.5) we obtain Expression

$$V_k = - \alpha_k \omega_k \left[\int_{\infty}^{0} e^{-\alpha_k s} q_1(s)\, ds + \int_{-\infty}^{0} e^{\alpha_k s} q_2(s)\, ds \right] \tag{3.6}$$

which is independent of h and coincides with the principal part of (3.4) when $\varepsilon \ll h$. This formula expresses the fact, natural from the physical point of view, of the existence of a difference of potential between the points of two arbitrary isolated noninteracting regions, in each of which the potential is determined by its "own" magnetic field.

The solution constructed in Section 2 can be considered in the neighborhood of the points $x = x_n$ as the result of the limiting transition of the type $R \to \infty$. In fact, the dimensionless resistance of the layer $(x_n - h, x_n + h)$

$$R = \int_{x_n - h}^{x_n + h} \psi^{-1}\, dx$$

for finite h and $\psi = \cos^2 \beta\xi$ is infinitely great because of the divergence of the integral.

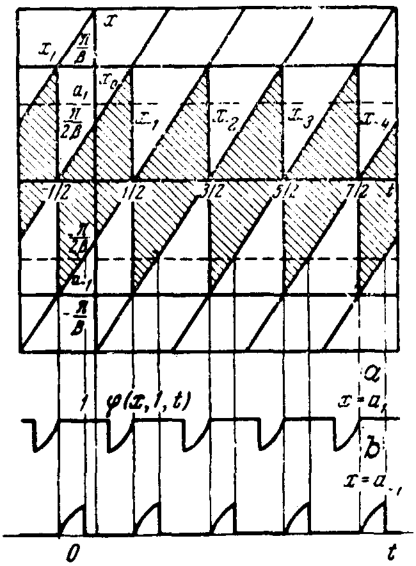

Fig. 3

From this it is to be concluded that the solution of Section 2 corresponds approximately to the distribution of conductivity

$$\psi = \begin{cases} \cos^2 \beta\xi & \text{for} \quad x \in [x_n + h,\ x_{n+1} - h] \\ \varepsilon & \text{for} \quad x \in [x_n - h,\ x_n + h] \end{cases}$$

and gives values of the potential in the regions $[x_n + h,\ x_{n+1} - h]$.

4. We consider in conclusion an example making use of the Formulas of Section 2. We shall assume that the magnetic field is given in the form of a step-function (4.1)

$$f(x) = \eta(x) = \begin{cases} 0 & \text{for} \quad x < 0, \quad f'(x) = \delta(x) \\ 1 & \text{for} \quad x > 0, \quad f''(x) = \delta'(x) \end{cases}$$

Suppose that as time t varies in the limits from $-\frac{1}{2}$ to $+\frac{1}{2}$, the point of discontinuity of the field $x = 0$ belongs to the segment $[x_{-1}, x_0]$. The location of the ends of the segment is moreover determined by Equations

$$x_{-1} = Vt - \frac{\pi}{2\beta}, \qquad x_0 = Vt + \frac{\pi}{2\beta}$$

Calculation by the formulas of Section 2 leads to the following expression for Φ_k when $|t| < \frac{1}{2}$:

$$\Phi_k = \begin{cases} 0 & \text{for} \quad x > x_0 \text{ and } x < x_{-1} \\ K^-[\gamma_k \cos \pi t \cosh \gamma_k (\pi t + {}^1\!/_2\pi) + \sin \pi t \sinh \gamma_k (\pi t + {}^1\!/_2\pi)] & \text{for} \quad x_{-1} < x < 0 \\ K^+[\gamma_k \cos \pi t \cosh \gamma_k (\pi t - {}^1\!/_2\pi) + \sin \pi t \sinh \gamma_k (\pi t - {}^1\!/_2\pi)] & \text{for} \quad 0 < x < x_0 \end{cases}$$

$$K^{\pm} = \frac{\omega_k}{\gamma_k \sinh \pi\gamma_k \cos \beta\xi} \sinh \gamma_k \left(\pi t \pm \frac{\pi}{2} - \beta x \right) \tag{4.2}$$

The true potential φ is identically equal to zero when $x < -\pi/\beta$ and when $x > \pi/\beta$ for all instants of time.

The meaning of this result is clear: for all values of t when $x < -\pi/\beta$ there is no magnetohydrodynamic interaction, whilst when $x > \pi/\beta$ there is no current, only a charge distribution. Currents can exist only inside that unique segment to which at a given instant belongs the point $x = 0$, where the field is included. The flow pattern of the currents does not differ qualitatively from the case of constant conductivity, i.e. the currents when $x > 0$ are directed on average against the electric field, and when $x < 0$ with the field, forming a closed loop.

In Fig.3*a*, the region in which φ varies periodically with respect to t for each x is depicted in the xt plane (a series of parallelograms). The corresponding curves are schematically shown in Fig.3*b*.

We note that φ varies continuously in the transition through the vertical sides of the parallelograms and vanishes at these points, whilst at the inclined sides φ undergoes discontinuities.

We shall now calculate the Joule dissipation in such a flow.

It is evident that from the usual formula

$$Q = \frac{U^2 B_0^2 \delta^2 \sigma_0}{c^2} \int_{-\infty}^{\infty} \int_{-1}^{1} \frac{\mathrm{j}^2}{\psi}\, dx\, dy \tag{4.3}$$

and so when ψ = const it follows that

$$Q = \frac{U^2 B_0^2 \delta^2 \sigma_0}{c^2} \int_{-\infty}^{\infty} \int_{-1}^{1} j_y f\, dx\, dy$$

Taking account of (1.4),(1.6),(4.1) and (4.2), we can moreover write

$$Q = \frac{2U^2 B_0^2 \delta^2 \sigma_0}{c^2} \sum_{k=1}^{\infty} (-1)^{k+1} \int_0^{x_0} \psi(\xi)\, \Phi_k(x, t)\, dx \tag{4.4}$$

Hence, substituting Expression (4.2) for Φ_k, we obtain

$$Q = \frac{U^2 B_0^2 \delta^2 \sigma_0 3}{c^2} \sum_{k=1}^{\infty} \frac{q_k(t)}{\alpha_k^4 \gamma_k \sinh \pi\gamma_k}, \qquad |t| < {}^1/_2 \tag{4.5}$$

$$q_k(t) = (\gamma_k^2 - 1)(\cos 2\pi t \cosh 2\gamma_k \pi t + \cosh \pi\gamma_k) +$$
$$+ (\gamma_k^2 + 1)(\cos 2\pi t \cosh \pi\gamma_k + \cosh 2\gamma_k \pi t) + 2\gamma_k \sin 2\pi t \sinh 2\gamma_k \pi t$$

If we now pass to the dimensional time $\pi\delta t/\beta U$ and subsequently let $\beta \to 0$, then in the limit we obtain the well known result for flow with constant conductivity [2]

$$Q = \frac{16 U^2 B_0^2 \delta^2 \sigma_0}{\pi^3 c^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \tag{4.6}$$

This limiting transition is valid only for $|t| < \frac{1}{2}$, since $q_k(\pm \frac{1}{2}) = 0$. Averaging over one period, the amount of dissipation is

$$Q^* = \int_{-1/2}^{1/2} Q dt = \frac{U^2 B_0^2 \delta^2 \sigma_0 3}{c^2} \sum_{k=1}^{\infty} \frac{q_k^*}{\alpha_k^4 \gamma_k \sinh \pi\gamma_k}$$
$$q_k^* = \frac{5\gamma_k^2 + 1}{\gamma_k^2 + 1} \frac{\sinh \pi\gamma_k}{\pi\gamma_k} + (\gamma_k^2 - 1)\cosh \pi\gamma_k \tag{4.7}$$

Because of the presence of zones with small conductivity, when $\beta \to 0$ the dissipation tends to a quantity equal to one half of (4.6).

In the other limiting case, for very large values of β, the following asymptotic formula holds:

$$Q^* \sim \frac{U^2 B_0^2 \delta^2 \sigma_0}{3\pi^4 c^2 \beta} (4\pi^2 - 15) \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \qquad (4.8)$$

The decrease of Q^* with increasing β has an obvious physical meaning, since then the region in which the currents flow is contracted proportionally to $1/\beta$ and the strength of the currents remains bounded.

We note that the value of the conductivity averaged over the whole channel $\sigma_* = \frac{1}{2}\sigma_0$ for all $\beta \neq 0$ and for all instants of time. The value of the dissipation Q_1 in the channel, calculated according to theory with constant conductivity, is therefore determined by Formula (4.6), where σ_0 must be replaced by σ_*. It is easy to see that in the general case $Q_1(\sigma_*) \neq Q^*(\beta)$, moreover the difference between these quantities becomes particularly significant if $\beta \gg 1$, when, according to (4.8), the quantity $Q^*(\beta)$ is very small.

The results obtained in Section 2.4 can be generalized to the case of the more complex law of variation of conductivity

$$\sigma = \sigma_{0k} \cos^2 \beta_k (x - Ut) \qquad \text{for } x - Ut \in [\xi_{k-1}, \xi_k] \quad (k = 0, \pm 1, \pm 2, \ldots)$$

where σ_{0k}, ξ_k and $\beta_k \geqslant 0$ are arbitrary numbers. If in the intervals $[\xi_{k-1}, \xi_k]$ the quantity $\cos \beta_k \xi$ does not vanish, then construction of a continuous solution is possible.

BIBLIOGRAPHY

1. Vatazhin, A.B. and Regirer, S.A., Priblizhennyi raschet raspredeleniia toka pri techenii provodiashchei zhidkosti po kanalu v magnitnom pole (Approximate calculation of the current distribution when a conducting fluid flows along a channel within a magnetic field). *PMM* Vol.26,№3, 1962.
2. Vatazhin, A.B., Nelotorye dvumernye zadachi o raspredelenii toka v elektroprovodnoi srede, dvizhushcheisia po kanalu v magnitnom pole (Certain two-dimensional problems concerning the current distribution in an electrically conducting medium moving along a channel in a magnetic field). PMTF, № 2, 1963.
3. Korsunskii, L.M., Elektromagnitnye gidrometricheskie pribory (Electro-Magnetic Hydrometric Instruments). Izd.Gos.Kom.standartov, mer i izmer. prib., 1964.
4. Iakubenko, A.E., II Vses. s"ezd po mekhanike. Annotatsii dokl. (All-Union Congress on Mechanics. Annotated proceedings). Izd."Nauka", 1964,p.239.
5. Fraidenraich, N., McGrath, I.A., Medin, S.A and Thring, M.W., A theoretical analysis of the thermodynamics of the striated magnetohydrodynamic system. J.appl.Phys., Vol.15, № 1, p.13-27, 1964.
6. Vatazhin, A.B. and Nemkova, N.G., Nekotorye dvumernye zadachi o raspredelenii elektricheskogo toka v kanale magnitnogidrodinamicheskogo generatora s neprovodiashchimi peregorodkami (Certain two-dimensional problems concerning the distribution of electric current in the channel of a magnetohydrodynamic generator with nonconducting partitions). PMTF, № 2, 1964.

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